Critical behavior of n-vector model with quenched randomness

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Abstract. We consider the Ginzburg–Landau phase transition model with $O(n)$ symmetry (i.e., the n–vector model) which includes a quenched randomness, i.e., a random temperature disorder. We have proven rigorously that within the diagrammatic perturbation theory the quenched randomness does not change the critical exponents at $n \to 0$, which is in contrast to the conventional point of view based on the perturbative renormalization group theory.

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1. Introduction

The phase transitions and critical phenomena is one of the most widely investigated topic in modern physics. Nevertheless, an eliminated number of exact and rigorous results are available, and they refer mainly to the two–dimensional systems [1] and fractals [2]. Rigorous results have been obtained also in four dimensions [3] based on an exact renormalization group (RG) technique [4]. The RG method, obviously, provides exact results at $d > 4$ (where d is the spatial dimensionality), but this case is somewhat trivial in view of critical phenomena. In three dimensions, approximate methods are usually used based on perturbation theory.

Here we present a particular result obtained within the diagrammatic perturbation theory. The Ginzburg–Landau phase transition model with $O(n)$ symmetry (i.e., the n–vector model) is considered, which includes a quenched random temperature disorder. The usual prediction of the perturbative RG field theory [5, 6, 7] is that, in the case of the spatial dimensionality $d < 4$ and small enough n (at $n = 1$ and $n \to 0$, in particular), the critical behavior of the n–component vector model is essentially changed by the quenched randomness. Here we challenge this conventional point of view based on a mathematical proof. We have proven rigorously that within the diagrammatic perturbation theory the critical exponents in the actually considered model cannot be changed by the quenched randomness at $n \to 0$.

2. The model

We consider a model with the Ginzburg–Landau Hamiltonian

$$H/T = \int \left[\left(r_0 + \sqrt{u}\, f(\mathbf{x}) \right) \varphi^2(\mathbf{x}) + c\, (\nabla \varphi(\mathbf{x}))^2 \right] d\mathbf{x}$$
$$+ uV^{-1} \sum_{i,j,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \varphi_i(\mathbf{k}_1)\varphi_i(\mathbf{k}_2)\, u_{\mathbf{k}_1+\mathbf{k}_2}\, \varphi_j(\mathbf{k}_3)\varphi_j(-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \qquad (1)$$

which includes a random temperature (or random mass) disorder represented by the term $\sqrt{u}\, f(\mathbf{x})\, \varphi^2(\mathbf{x})$. For convenience, we call this model the random model. In Eq. (1) $\varphi(\mathbf{x})$ is an n–component vector (the order parameter field) with components $\varphi_i(\mathbf{x}) = V^{-1/2} \sum_{\mathbf{k}} \varphi_i(\mathbf{k}) e^{i\mathbf{kx}}$, depending on the coordinate $\mathbf{x}$, and $f(\mathbf{x}) = V^{-1/2} \sum_{\mathbf{k}} f_{\mathbf{k}} e^{i\mathbf{kx}}$ is a random variable with the Fourier components $f_{\mathbf{k}} = V^{-1/2} \int f(\mathbf{x}) e^{-i\mathbf{kx}} d\mathbf{x}$. Here V is the volume of the system, T is the temperature, and $\varphi_i(\mathbf{k})$ is the Fourier transform of $\varphi_i(\mathbf{x})$. An upper limit of the magnitude of wave vector k_0 is fixed. It means that the only allowed configurations of the order parameter field are those corresponding to $\varphi_i(\mathbf{k}) = 0$ at $k > k_0$. This is the limiting case $m \to \infty$ (m is integer) of the model where all configurations of $\varphi(\mathbf{x})$ are allowed, but Hamiltonian (1) is completed by term $\sum\limits_{i,\mathbf{k}} (k/k_0)^{2m} |\, \varphi_i(\mathbf{k})\, |^2$.

The perturbation expansions of various physical quantities in powers of the coupling constant u are of interest. In this case n may be considered as a continuous parameter. In particular, the case $n \to 0$ has a physical meaning describing the statistics of polymers [7].

The system is characterized by the two–point correlation function $G_i(\mathbf{k})$ defined by the equation

$$\langle \varphi_i(\mathbf{k}) \varphi_j(-\mathbf{k}) \rangle = \delta_{i,j}\, G_i(\mathbf{k}) \; . \qquad (2)$$

Because of the $O(n)$ symmetry of the considered model, we have $G_i(\mathbf{k}) \equiv G(\mathbf{k})$, i.e., the index i may be omitted. It is supposed that the averaging is performed over the $f(\mathbf{x})$ configurations with a Gaussian distribution for the Fourier components $f_{\mathbf{k}}$, i.e., the configuration $\{f_{\mathbf{k}}\}$ is taken with the weight function

$$P(\{f_{\mathbf{k}}\}) = Z_1^{-1} \exp\left(- \sum_{\mathbf{k}} b(\mathbf{k}) \mid f_{\mathbf{k}} \mid^2 \right) \; , \qquad (3)$$

where

$$Z_1 = \int \exp\left(- \sum_{\mathbf{k}} b(\mathbf{k}) \mid f_{\mathbf{k}} \mid^2 \right) D(f) \; , \qquad (4)$$

and $b(\mathbf{k})$ is a positively defined function of k. Eq. (1) defines the simplest random model considered in [6] (according to the universality hypothesis, the factor $\sqrt{u}$ does not make an important difference). Our random model describes a quenched randomness since the distribution over the configurations $\{f_{\mathbf{k}}\}$ of the random variable is given (by Eqs. (3) and (4)) and does not depend neither on temperature nor the configuration $\{\varphi_i(\mathbf{k})\}$ of the

order parameter field. More precisely, the common distribution over the configurations $\{f_{\mathbf{k}}\}$ and $\{\varphi_i(\mathbf{k})\}$ is given by

$$P(\{f_{\mathbf{k}}\},\{\varphi_i(\mathbf{k})\}) = P(\{f_{\mathbf{k}}\}) \times Z_2^{-1}(\{f_{\mathbf{k}}\}) \exp(-H/T)\,, \tag{5}$$

where $Z_2(\{f_{\mathbf{k}}\}) = \int \exp(-H/T) D(\varphi)$ and H is defined by Eq. (1). For comparison, the common distribution is the equilibrium Gibbs distribution in a case of annealed randomness sometimes considered in literature.

3. A basic theorem

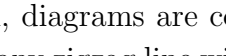
We have proven the following theorem.

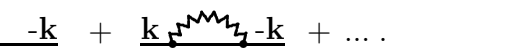
Theorem. In the limit $n \to 0$, the perturbation expansion of the correlation function $G(\mathbf{k})$ in u power series for the random model with the Hamiltonian (1) is identical to the perturbation expansion for the corresponding model with the Hamiltonian

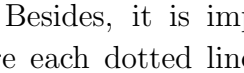
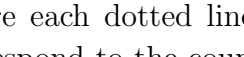
$$\begin{aligned} H/T = & \int \left[r_0\, \varphi^2(\mathbf{x}) + c\,(\nabla\varphi(\mathbf{x}))^2 \right] d\mathbf{x} \\ & + uV^{-1} \sum_{i,j,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \varphi_i(\mathbf{k}_1)\varphi_i(\mathbf{k}_2)\, \tilde{u}_{\mathbf{k}_1+\mathbf{k}_2}\, \varphi_j(\mathbf{k}_3)\varphi_j(-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \end{aligned} \tag{6}$$

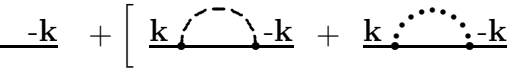
where $\tilde{u}_{\mathbf{k}} = u_{\mathbf{k}} - \frac{1}{2}\left\langle |\, f_{\mathbf{k}}\,|^2 \right\rangle$.

For convenience, we call the model without the term $\sqrt{u}\, f(\mathbf{x})\, \varphi^2(\mathbf{x})$ the pure model, since this term simulates the effect of random impurities [6].

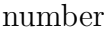
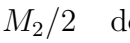
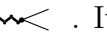
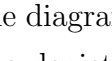
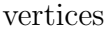
Proof of the theorem. According to the rules of the diagram technique, the formal expansion for $G(\mathbf{k})$ involves all connected diagrams with two fixed outer solid lines. In the case of the pure model, diagrams are constructed of the vertices  , with factor $-uV^{-1}\tilde{u}_{\mathbf{k}}$ related to any zigzag line with wave vector $\mathbf{k}$. The solid lines are related to the correlation function in the Gaussian approximation $G_0(\mathbf{k}) = 1/\left(2r_0 + 2ck^2\right)$. Summation over the components $\varphi_i(\mathbf{k})$ of the vector $\varphi(\mathbf{k})$ yields factor n corresponding to each closed loop of solid lines in the diagrams. According to this, the formal perturbation expansion is defined at arbitrary n. In the limit $n \to 0$, all diagrams of $G(\mathbf{k})$ vanish except those which do not contain the closed loops. In such a way, for the pure model we obtain the expansion

$G(\mathbf{k}) =$  $+ \ldots$. (7)

In the case of the random model, the diagrams are constructed of the vertices  and  . Besides, it is important that only those diagrams give a nonzero contribution where each dotted line is coupled to another dotted line. The factors $uV^{-1}\left\langle |\, f_{\mathbf{k}}\,|^2 \right\rangle$ correspond to the coupled dotted lines and the factors $-uV^{-1}u_{\mathbf{k}}$ correspond to the dashed lines. Thus, we have

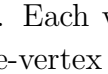
$G(\mathbf{k}) =$  $+ \ldots$. (8)

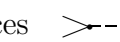
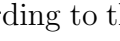
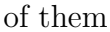
In our notation the combinatorial factor corresponding to any specific diagram is not given explicitly, but is implied in the diagram itself. In the random model, first the correlation function $G(\mathbf{k})$ is calculated at a fixed $\{f_{\mathbf{k}}\}$ according to the distribution $Z_2^{-1}(\{f_{\mathbf{k}}\})\exp(-H/T)$ (which corresponds to diagrams where solid lines are coupled, but the dotted lines with factors $-\sqrt{u}\,V^{-1/2}f_{\mathbf{k}}$ are not coupled), performing the averaging with the weight (3) over the configurations of the random variable (i.e., the coupling of the dotted lines) afterwards. According to this procedure, the diagrams of the random model in general (not only at $n \to 0$) do not contain parts like  ,  ,  , etc. Such parts would appear after the coupling of dotted lines only if unconnected (i.e., consisting of separate parts) diagrams with fixed $\{f_{\mathbf{k}}\}$ would be considered.

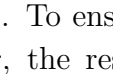
Thus, in the considered random model, the term of the l–th order in the perturbation expansion of $G(\mathbf{k})$ in u power series is represented by diagrams constructed of a number M_1 of vertices  and an even number M_2 of vertices  (i.e., $M_2/2$ double–vertices ), such that $l = M_1 + M_2/2$. In the pure model, defined by Eq. (6), this term is represented by diagrams constructed of l vertices  . It is clear and evident from Eqs. (7) and (8) that all diagrams of the random model are obtained from those of the pure model if any of the zigzag lines is replaced either by a dashed or by a dotted line, performing summation over all such possibilities. Note that such a method is valid in the limit $n \to 0$, but not in general. The problem is that, except the case $n \to 0$, the diagrams of the pure model contain parts like  ,  ,  , etc. If all the depicted here zigzag lines are replaced by the dotted lines, then we obtain diagrams which are not allowed in the random model, as it has been explained before. At $n \to 0$, the only problem is to determine the combinatorial factors for the diagrams obtained by the above replacements. For a diagram constructed of M_1 vertices  and M_2 vertices  the combinatorial factor is the number of possible different couplings of lines, corresponding to the given topological picture, divided by $M_1!M_2!$.

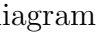
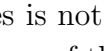
It is suitable to make some systematic grouping of the diagrams of the random model. The following consideration is valid not only for the diagrams of the two–point correlation function, but also for free energy diagrams (connected diagrams without outer lines) and for the diagrams of $2m$–point correlation function (i.e., the diagrams with $2m$ fixed outer solid lines, containing no separate parts unconnected to these lines). It is supposed that at $n \to 0$ the main terms are retained, which means that the free energy diagrams contain a single loop of solid lines. We define that all diagrams which can be obtained from the i–th diagram (i.e., the diagram of the i–th topology) of the pure model, belong to the i–th group. The sum of the diagrams of the i–th group can be found by the following algorithm.

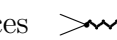
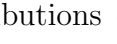
1. First, the i–th diagram of pure model is depicted in an a priori defined way.

2. Each vertex  is replaced either by the vertex  , or by the double-vertex , performing the summation over all possibilities. Besides, all

vertices  and  and all lines are numbered before coupling, and all the distributions of the numbered vertices and lines over the numbered positions (arranged according to the given picture defined in step 1 and according to the actually considered choice, defining which of the vertices  must be replaced by  and which of them must be replaced by ) are counted as different. Each specific realization is summed over with the weight $1/(M_1!M_2!)$.

3. To ensure that each specific diagram is counted with the correct combinatorial factor, the result of summation in step 2 is divided by the number of independent symmetry transformations (including the identical transformation) S_i for the considered i–th diagram constructed of vertices  , where the symmetry transformation of a diagram is defined as any possible redistribution (such that the outer solid lines are fixed) of vertices and coupled lines not changing the given picture. Really, the coupling of lines is not changed if any of the symmetry transformations with any of the specific diagrams of the i–th group is performed, whereas, according to the algorithm of step 2, original and transformed diagrams are counted as different.

It is convenient to modify step 2 as follows. Choose any one replacement of the vertices  by  and  , and perform the summation over all such possibilities. For any specific choice we consider only one of the possible $M_1!M_2!$ distributions of the numbered M_1 vertices  and M_2 vertices  over the fixed numbered positions, and make the summation with weight 1 instead of the summation over $M_1!M_2!$ equivalent (i.e., equally contributing) distributions with the weight $1/(M_1!M_2!)$.

Note that the location of any vertex  is defined by fixing the position of dashed line, the orientation of which is not fixed. According to this, the summation over all possible distributions of lines (numbered before coupling) for one fixed location of vertices (as consistent with the modified step 2) yields factor $8^{M_1}4^{M_2/2}$. The i–th diagram of the pure model also can be calculated by such an algorithm. In this case the summation over all possible line distributions yields a factor of 8^l, where $l = M_1 + M_2/2$ is the total number of vertices  in the i–th diagram. Obviously, the summation of diagrams of the i–th group can be performed with factors 8^l instead of $8^{M_1}4^{M_2/2}$, but in this case factors $\frac{1}{2}uV^{-1}\left\langle \mid f_{\mathbf{k}} \mid^2 \right\rangle$ must be related to the coupled dotted lines instead of $uV^{-1}\left\langle \mid f_{\mathbf{k}} \mid^2 \right\rangle$. In this case the summation over all possibilities where zigzag lines are replaced by dashed lines with factors $-uV^{-1}u_{\mathbf{k}}$ and by dotted lines with factors $\frac{1}{2}uV^{-1}\left\langle \mid f_{\mathbf{k}} \mid^2 \right\rangle$, obviously, yields a factor $uV^{-1}\left(-u_{\mathbf{k}} + \frac{1}{2}\left\langle \mid f_{\mathbf{k}} \mid^2 \right\rangle\right) \equiv -uV^{-1}\tilde{u}_{\mathbf{k}}$ corresponding to each zigzag line with wave vector $\mathbf{k}$. Thus, the sum over the diagrams of the i–th group is identical to the i–th diagram of the pure model defined by Eq. (6). By this the theorem has proved.

4. Remarks and conclusions

The theorem has been formulated for the two–point correlation function, but the proof, in fact, is more general, as regards the relation between diagrams of random and pure

models. Thus, the statement of the theorem is true also for free energy and for $2m$–point correlation function.

According to the proven theorem and this remark, at $n \to 0$ the considered pure and random models cannot be distinguished within the diagrammatic perturbation theory. Thus, if, in principle, critical exponents can be determined from the diagrammatic perturbation theory at $n \to 0$, then, in this limit, the critical exponents for the random model are the same as for the pure model. We think that in reality correct values of critical exponents can be determined from the diagrammatic perturbation theory, therefore the quenched random temperature disorder does not change the universality class at $n \to 0$. This our conclusion correlates with results of some other authors. In particular, there is a good evidence that the universality class is not changed by the quenched randomness at $n = 1$. It has been shown by extensive Monte–Carlo simulations of two–dimensional dilute Ising models [8] that the critical exponent of the defect magnetization is a continuous function of the defect coupling. By analyzing the stability conditions, it has been concluded in Ref. [8] that the critical exponent ν of the bulk correlation length of the random Ising model does not depend on the dilution, i.e., $\nu = 1$ at $d = 2$ both for diluted and not diluted Ising models. The standard (pertubative) RG method predicts the change of the universality class by the quenched randomness. We think, this is a false prediction. The fact that the standard RG method provides incorrect result is not surprising, since it has been demonstrated (in fact, proven) in Ref. [9] that this method is not valid at $d < 4$.

References

[1]Rodney J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London etc.,1982)
[2]Jose Arnaldo Redinz, Aglae C. N. de Magelhaes, *Phys. Rev. B* **51**, 2930 (1995)
[3]T. Hara, H. Tasaki, *J. Stat. Phys.* **47**, 99 (1987)
[4]K. Gawedzki, A. Kupiainen, *Commun. Math. Phys.* **99**, 197 (1985)
[5]K.G. Wilson, M.E. Fisher, *Phys.Rev.Lett.* **28**, 240 (1972)
[6]Shang–Keng Ma, *Modern Theory of Critical Phenomena* (W.A. Benjamin, Inc., New York, 1976)
[7]J. Zinn–Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, 1996)
[8]F. Szalma and F. Igloi, *Abstracts MECO 24, P69*
(March 8-10, Lutterstadt Wittenberg, Germany, 1999)
[9]J. Kaupužs, *archived as cond-mat/0001414 of xxx.lanl.gov archive*, January 2000